Oscillating Airfoils: Part II. Newtonian Flow Theory and Application to Power-Law Bodies in Hypersonic Flow

Ronald M. Barron*
University of Windsor, Windsor, Ontario, Canada
and
P. Mandl†
Carleton University, Ottawa, Ontario, Canada

A theoretical investigation of the hypersonic flow past two-dimensional airfoils oscillating with arbitrary frequency is undertaken. An unsteady Newtonian flow theory is developed which includes the contributions due to centrifugal force by considering gasdynamic theory in the limit $\gamma \to 1$ and $M_\infty \to \infty$. The theory is valid for both sharp- and blunt-nosed curved bodies, provided the shock wave remains attached at the nose. The case of airfoils supporting power-law shocks ($y = x^m$) and pitching about a pivot on its axis of symmetry is examined in light of this theory. It is seen that pivot positions ahead of $h = (3m-1)^2/9m^2$ tend to destabilize the airfoil, while those behind it have a stabilizing effect.

I. Introduction

EARLY work in the area of hypersonic flow past oscillating airfoils neglected the effects of the waves generated by the body motion and reflected from the bow shock wave. In his doctoral dissertation, McIntosh[1] treated this aspect of the problem for the thin wedge, for which one need not consider the full equations of motion, since hypersonic small-disturbance theory gives a good approximation to the complete flow. However, due mainly to manufacturing problems and the extremely high temperatures attained in hypersonic flight, hypersonic vehicles will have blunt noses, although probably slendering out at a short distance from the nose. The mathematical consequence of the blunt nose is to introduce a singularity in the hypersonic small-disturbance equations so that they are no longer valid in the entire shock layer. The streamlines that pass through the nearly normal portion of the shock in the nose region form a layer of relatively high entropy air enveloping the body surface.[2] This layer is a region of nonuniformity characteristic of singular perturbation problems. Taking this into account, any unsteady theory for blunt-nosed (or thick, sharp-nosed) oscillating airfoils must begin from the full equations of motion.

The purpose of the present investigation is to account for the effect of the blunt nose on oscillating hypersonic aircraft. The mathematical formulation is as in Part I, Secs. II and III.[3] The equations governing the perturbed flow do not permit a solution for arbitrary body shape due to their complexity. Hence, in an effort to extract an analytical solution, the Newtonian approximation ($\gamma \to 1$, $M_\infty \to \infty$) is introduced, since it will not neglect the effects of the blunt nose. In the Newtonian limit, the shock collapses to the body surface and hence a stretching of the coordinate system is necessary for an examination of the flowfield in the shock layer.

The theory is applied to airfoils supporting power-law shocks and oscillating with arbitrary frequency in pitch about a pivot arbitrarily located on its axis of symmetry. Restoring- and damping-moment coefficients are obtained in Sec. IVB and the effect of pivot position on the stability of the airfoil is discussed in Sec. IVC.

II. Newtonian Approximation

A fundamental small parameter for the Newtonian limit is defined by

$$\epsilon = (\gamma - 1)/(\gamma + 1) \tag{1}$$

Conservation of mass requires that the shock wave lie a distance $0(\epsilon)$ off the body. However, the assumption of shock attachment at the nose is retained and hence the subsequent theory is not valid in the nose region. To preserve the boundary conditions on both the shock and body in the limit $\epsilon \to 0$ and $M_\infty \to \infty$, it is necessary to stretch the coordinate system by measuring distances across the shock layer relative to ϵ.[4] Examining the usual steady-state Newtonian flow solutions, the following expansions, with all variables in nondimensional form, are assumed:

$$\begin{aligned} \rho_0(x,\psi;\gamma,M_\infty) &= (1/\epsilon)\rho_n(x,\psi_n;N) + \ldots \\ u_0(x,\psi;\gamma,M_\infty) &= u_n(x,\psi_n;N) + \ldots \\ v_0^*(x,\psi;\gamma,M_\infty) &= \epsilon v_n(x,\psi_n;N) + \ldots \\ p_0(x,\psi;\gamma,M_\infty) &= p_n(x,\psi_n;N) + \ldots \end{aligned} \tag{2}$$

where $N \equiv 1/(\epsilon M_\infty^2)$ remains finite as $\epsilon \to 0$ and $M_\infty \to \infty$. In these expansions, a Von Mises transformation has been made from (x,y) to (x,ψ) and the quantity ψ_n is defined through the relation

$$\psi_n = \psi_{0n} - y_{s0}(x) + 0(\epsilon)$$

where ψ_{0n} is the stream function associated with the first-order steady-state Newtonian flowfield.

Again, we assume that the body is exhibiting small harmonic oscillations which cause only small disturbances to the mean steady flow. As in Part I, all flow variables $F(x,y,t)$ are expanded in powers of the amplitude, α, of the oscillations; that is,

$$F(x,y,t) = F_0(x,y) + \alpha e^{ikt} F_1(x,y) + 0(\alpha^2)$$

with subscripts 0 and 1 referring to the mean steady flow and

Received Sept. 2, 1977; revision received Feb. 28, 1978.

Index categories: Shock Waves and Detonations; Supersonic and Hypersonic Flow; Nonsteady Aerodynamics.

*Assistant Professor, Dept. of Mathematics. Member AIAA.
†Professor, Dept. of Mathematics.

the first-order perturbed flow, respectively. The Newtonian limit is such that the parameters ϵ and α may be considered independent. This implies that the "unsteadiness" of the flow is not affected by the Newtonian limit $\epsilon \to 0$, $M_\infty \to \infty$, and vice-versa. Hence, it is proposed to expand the first-order perturbed flow quantities as:

$$\rho_1(x,\psi;\gamma,M_\infty) = (1/\epsilon)R(x,\psi_n;N) + \dots$$
$$u_1(x,\psi;\gamma,M_\infty) = U(x,\psi_n;N) + \dots$$
$$v_1^*(x,\psi;\gamma,M_\infty) = \epsilon V(x,\psi_n;N) + \dots$$
$$p_1(x,\psi;\gamma,M_\infty) = P(x,\psi_n;N) + \dots \qquad (3)$$

The oscillating shock shape is defined by the function $y_s(x,t)$, which may be expanded as:

$$y_s(x,t) = y_{s0}(x) - \alpha e^{ikt} f_s(x) + 0(\alpha^2) = y_{b0}(x) + \epsilon y_{sn}(x) - \alpha e^{ikt}\{f_b(x) + \epsilon f_{sn}(x)\} + 0(\epsilon^2,\alpha^2) \qquad (4)$$

Since the function y_{sn} may be determined in the steady flow and f_b is prescribed, the function f_{sn} is the only unknown in this equation which must be determined by the first-order perturbed flow problem.

With x and ψ_n as the independent variables, the mean shock location is given by $\psi_n = 0$ and the mean airfoil surface has the equation $\psi_n = -y_{s0}(x) = -y_{b0}(x) + 0(\epsilon)$. Also, it should be noted that, as is inherent in the Newtonian flow approximation, the theory developed here is not valid beyond the point where the steady Newtonian pressure p_n falls to zero.

III. Unsteady Newtonian Flow Solution

A. First-Order Equations and Boundary Conditions

Applying expansions (2) and (3), to Eq. (20) in Part I, we obtain a system of equations which represent the Newtonian approximation to the first-order perturbed flow equations. In matrix form, we have

$$M_{1n}\frac{\partial W_n}{\partial x} + M_{2n}\frac{\partial W_n}{\partial \psi_n} + M_{3n}W_n = M_{4n} \qquad (5)$$

where

$$W_n = \begin{bmatrix} R \\ U \\ V \\ P \end{bmatrix} \qquad M_{1n} = \begin{bmatrix} u_n & \rho_n & 0 & 0 \\ 0 & u_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & p_n & 0 & u_n \end{bmatrix}$$

$$M_{2n} = \begin{bmatrix} -y'_{s0}u_n & -\rho_n(y'_{s0}+\rho_n v_n) & \rho_n^2 u_n & 0 \\ 0 & -y'_{s0}u_n & 0 & -y'_{s0}u_n \\ 0 & 0 & 0 & u_n \\ 0 & -p_n(y'_{s0}+\rho_n v_n) & \rho_n p_n u_n & -y'_{s0}u_n \end{bmatrix}$$

$$M_{4n} = \begin{bmatrix} 0 \\ -u_n f'_b(x)\partial p_n/\partial\psi_n \\ (y_{s0}'^2+1)^{-1}\{u_n^2 f''_b(x) + u_n[2ik + y'_{s0}\partial p_n/\partial\psi_n + D_n u_n]f'_b(x) - k^2 f_b(x)\} \\ 0 \end{bmatrix}$$

and $M_{3n} = (m_{ij})$, with

$$m_{11} = ik + D_n u_n - \rho_n v_n \frac{\partial u_n}{\partial \psi_n} + \rho_n u_n \frac{\partial v_n}{\partial \psi_n}$$

$$m_{12} = D_n \rho_n - \rho_n v_n \frac{\partial \rho_n}{\partial \psi_n}, \quad m_{13} = \rho_n u_n \frac{\partial \rho_n}{\partial \psi_n}$$

$$m_{21} = \left(\frac{y'_{s0}u_n}{\rho_n}\right)\frac{\partial p_n}{\partial \psi_n}, \quad m_{22} = ik + D_n u_n - \rho_n v_n \frac{\partial u_n}{\partial \psi_n}$$

$$m_{23} = \rho_n u_n \frac{\partial u_n}{\partial \psi_n}, \quad m_{31} = -\left(\frac{u_n}{\rho_n}\right)\frac{\partial p_n}{\partial \psi_n}$$

$$m_{32} = 2y''_{s0}u_n\{y_{s0}'^2+1\}^{-1}, \quad m_{42} = D_n p_n - \rho_n v_n \frac{\partial p_n}{\partial \psi_n}$$

$$m_{43} = \rho_n u_n \frac{\partial p_n}{\partial \psi_n}, \quad m_{44} = ik - \left(\frac{u_n}{\rho_n}\right)D_n \rho_n$$

$$m_{14} = m_{24} = m_{33} = m_{34} = m_{41} = 0 \qquad (6)$$

In Eqs. (6), the operator D_n is defined as $D_n \equiv (\partial/\partial x) - y'_{s0}(\partial/\partial\psi_n)$.

The boundary conditions associated with Eq. (5) are those at the body surface and at the shock. In order for the relative normal velocity to vanish at the surface it is required that:

$$V(x,\psi_n) = 0 \text{ at } \psi_n = -y_{s0}(x) \qquad (7)$$

The conditions at the shock wave are[5]:

$$R(x,\psi_n)\Big|_{\psi_n=0} = \rho_n(x,0)u_n(x,0)\frac{\partial \rho_n}{\partial \psi_n}(x,0)f_{sn}(x) - \frac{2y'_{b0}(x)N\{f'_b(x) + ik[y_{s0}'^2(x)+1]f_b(x)\}}{\{N[y_{s0}'^2(x)+1] + y_{s0}'^2(x)\}^2}$$

$$U(x,\psi_n)\Big|_{\psi_n=0} = \rho_n(x,0)u_n(x,0)\frac{\partial u_n}{\partial \psi_n}(x,0)f_{sn}(x) + \frac{2y'_{s0}(x)f'_b(x)}{[y_{s0}'^2(x)+1]^2} + \frac{iky'_{s0}(x)f_b(x)}{y_{s0}'^2(x)+1}$$

$$V(x,\psi_n)\Big|_{\psi_n=0} = \rho_n(x,0)u_n(x,0)\frac{\partial v_n}{\partial \psi_n}(x,0)f_{sn}(x) + f'_b(x) + \frac{2y'_{s0}(x)y'_{sn}(x)f'_b(x)}{(y_{s0}'^2(x)+1)^2} - \frac{f'_{sn}(x)}{y_{s0}'^2(x)+1} + \frac{iky'_{s0}(x)y'_{sn}(x)f_b(x)}{y_{s0}'^2(x)+1} + ikf_b(x) - ikf_{sn}(x)$$

$$P(x,\psi_n)\Big|_{\psi_n=0} = \rho_n(x,0)u_n(x,0)\frac{\partial p_n}{\partial \psi_n}(x,0)f_{sn}(x) - \frac{2y'_{s0}(x)}{[y_{s0}'^2(x)+1]}\{f'_b(x) + ik[y_{s0}'^2(x)+1]f_b(x)\} \qquad (8)$$

To complete the formulation, the condition of shock attachment at the nose,

$$f_{sn}(0) = 0 \qquad (9)$$

must be included. In these equations we have replaced $y_{b0}(x)$ by $y_{s0}(x) - \epsilon y_{sn}(x)$ [see Eq. (4)], since most existing analytical solutions to the steady flow problem are solutions of the inverse problem; that is, the shock shape is assumed known and the corresponding body shape is derived as part of the solution.

B. Canonical System

Proceeding as in Part I, we find that det $M_{2n} = -\rho_n p_n u_n^4 y_{s0}'^2$. Assuming $\rho_n, p_n, u_n, y_{s0}'$ are nonzero, Eq. (5) may be written as:

$$\frac{\partial W_n}{\partial \psi_n} + A_n \frac{\partial W_n}{\partial x} + M_{2n}^{-1} M_{3n} W_n = M_{2n}^{-1} M_{4n} \tag{10}$$

where $A_n = M_{2n}^{-1} M_{1n}$. The matrix A_n has two distinct eigenvalues, each of multiplicity two,

$$\lambda_1 = \lambda_2 = 0$$

$$\lambda_3 = \lambda_4 = -1/y_{s0}'(x) \tag{11}$$

From these it is possible to obtain only three linearly independent eigenvectors. Hence, there is no matrix which diagonalizes A_n. However, using the concepts of Jordan blocks and generalized eigenvectors,[6] one can find a nonsingular matrix

$$Q = \begin{bmatrix} 0 & \rho_n^2 & 1 & 1 \\ 0 & 0 & u_n & 0 \\ 1 & 0 & v_n & 0 \\ 0 & \rho_n p_n & 0 & 0 \end{bmatrix}$$

such that

$$Q^{-1} A_n Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/y_{s0}'(x) & 0 \\ 0 & 0 & 0 & -1/y_{s0}'(x) \end{bmatrix} \tag{12}$$

If one defines $Z = (Z_i) = Q^{-1} W_n$

$$= \begin{bmatrix} -v_n U/u_n + V \\ P/(\rho_n p_n) \\ U/u_n \\ R - U/u_n - \rho_n P/p_n \end{bmatrix} \tag{13}$$

Eq. (10) takes the canonical form

$$\frac{\partial Z}{\partial \psi_n} + J \frac{\partial Z}{\partial x} + KZ = G \tag{14}$$

where

$$J = Q^{-1} A_n Q$$

$$K = (k_{ij}) = Q^{-1} A_n \frac{\partial Q}{\partial x} + Q^{-1} \frac{\partial Q}{\partial \psi_n} + Q^{-1} M_{2n}^{-1} M_{3n} Q$$

$$G = (G_i) = Q^{-1} M_{2n}^{-1} M_{4n} \tag{15}$$

Boundary condition (7) at the body surface becomes

$$Z_1(x, -y_{s0}(x)) = 0 \tag{16}$$

The shock conditions associated with system (14) can be easily obtained from Eqs. (13) and (8), and hence values for $Z_i(x,0)$ are known in terms of the steady-state quantities, the known function f_b, and the unknown function f_{sn}. Expressions for $Z_i(x,0)$, k_{ij}, and G_i can be found in Ref. 5.

C. Solution of Perturbed Flow Problem for Arbitrary Body Shape

The system of equations (14) can be solved by transforming it to a system of integral equations and applying an iterative method of solution. As in Part I, we integrate Eq. (14) along the characteristics which, through any point (ξ,η) in the (x,ψ_n) plane, are given by:

$$x = X_k(\psi_n;\xi,\eta) = \begin{cases} \xi & k=1,2 \\ y_{s0}'^{-1}(\eta + y_{s0}(x) - \psi_n) & k=3,4 \end{cases} \tag{17}$$

Treating ψ_n as a parameter along the characteristics, Eq. (14) is equivalent to the system of integral equations

$$Z_1(\xi,\eta) = Z_1(X_1(0;\xi,\eta),0) - \int_0^\eta \frac{\partial Z_2}{\partial x}(X_1(\sigma;\xi,\eta),\sigma)\,d\sigma - \sum_{i=1}^4 \int_0^\eta k_{1i} Z_i(X_1(\sigma;\xi,\eta),\sigma)\,d\sigma + \int_0^\eta G_1(X_1(\sigma;\xi,\eta),\sigma)\,d\sigma \tag{18a}$$

and for $\nu = 2, 3, 4$:

$$Z_\nu(\xi,\eta) = Z_\nu(X_\nu(0;\xi,\eta),0) - \sum_{i=1}^4 \int_0^\eta k_{\nu i} Z_i(X_\nu(\sigma;\xi,\eta),\sigma)\,d\sigma + \int_0^\eta G_\nu(X_\nu(\sigma;\xi,\eta),\sigma)\,d\sigma \tag{18b}$$

Following Garabedian,[7] define a sequence of successive approximations $Z_\nu^{(p)}$, $\nu = 1, 2, 3, 4$ for the solution of the set of integral equations, Eqs. (18), by

$$Z_\nu^{(0)}(\xi,\eta) = Z_\nu(\xi,0) \tag{19a}$$

$$Z_1^{(p+1)}(\xi,\eta) = Z_1(\xi,0) - \int_0^\eta \frac{\partial Z_2^{(p+1)}}{\partial x}(\xi,\sigma)\,d\sigma - \sum_{i=1}^4 \int_0^\eta k_{1i} Z_i^{(p)}(\xi,\sigma)\,d\sigma + \int_0^\eta G_1(\xi,\sigma)\,d\sigma \tag{19b}$$

and for $\nu = 2, 3, 4$:

$$Z_\nu^{(p+1)}(\xi,\eta) = Z_\nu(\xi,0) - \sum_{i=1}^4 \int_0^\eta k_{\nu i} Z_i^{(p)}(X_\nu(\sigma;\xi,\eta),\sigma)\,d\sigma + \int_0^\eta G_\nu(X_\nu(\sigma;\xi,\eta),\sigma)\,d\sigma \tag{19c}$$

The second term on the right-hand side of Eq. (19b) requires computation of $Z_2^{(p+1)}$ prior to that of $Z_1^{(p+1)}$. Equation (13) shows that the pressure field is given by the sequence $Z_2^{(p)}$ calculated using Eq. (19c). However, this yields expressions for $Z_2^{(p)}$ in terms of the unknown function f_{sn} because of the conditions at the shock, i.e., $Z_\nu(\xi,0)$ depends upon f_{sn} [cf. comments after Eq. (16)]. The function f_{sn} can be determined by calculating the first approximation to Z_1 and applying boundary condition, Eq. (16). This may be represented by [cf., Eq. (18a)]

$$Z_1(\xi,0) - \int_0^{-y_{s0}(\xi)} \frac{\partial Z_2^{(1)}}{\partial x}(\xi,\sigma)\,d\sigma - \sum_{i=1}^4 Z_i(\xi,0) \int_0^{-y_{s0}(\xi)} k_{1i}(\xi,\sigma)\,d\sigma + \int_0^{-y_{s0}(\xi)} G_1(\xi,\sigma)\,d\sigma = 0 \tag{20}$$

Evaluating the integrals in Eq. (20) leads to a first-order, linear, ordinary differential equation for $f_{sn}(\xi)$, with coefficients depending on the steady-state functions $u_n, v_n, \rho_n, p_n, y_{s0}$ and their derivatives, as well as the mode shape of the oscillations, f_b.[5] In general, the coefficients are complex and f_{sn} is a complex-valued function of a real variable.

The theory developed here is applicable to any body profile (provided y''_{b0} exists) for which the shock remains attached at the nose. In order to evaluate the integrals in Eqs. (19) and (20), it is necessary to have analytical solutions for the steady Newtonian flow problem. We now proceed to apply the results of this section to airfoils supporting power-law shocks.

IV. Application to Power-Law Bodies

Barron[8] has obtained asymptotic expansions, valid for large distances downstream, for the mean steady flow variables u_n, v_n, p_n, ρ_n, and y_{sn} when the shock shape is described by $y_{s0}(x) = x^m$, with $N=0$ and $2/3 < m < 4/5$. The results given by Eqs. (19) of Ref. 8 can be recast in our present notation by replacing ψ by ψ_{0n}, and hence are not repeated here.‡

We consider the case of a rigid power-law airfoil oscillating with small amplitude about an arbitrary pivot located along the chord line at a distance $x=h$ from the leading edge, and hence take

$$f_b(x) = x - h$$

Of primary importance are the unsteady forces and moments on the airfoil and hence the pressure on the body surface must be determined.

A. Surface Pressure

Restricting ourselves to the first iterate, we evaluate Eq. (19c) with $p=0$ and $\nu=2$ [cf. Eq. (13)]. The calculations are lengthy and tedious, the details of which may be found in Ref. 5. Equation (19c) yields

$$\begin{aligned}
Z_2^{(1)}(\xi,\eta) = {} & Z_2(\xi,0)\{1+\ln[m^2+\xi^{2(1-m)}] - \ln\left[\frac{2m-1}{m}+\frac{1-m}{m}\frac{\eta+\xi^{2m}}{\xi^m}\right] - \ln[m^2+(\eta+\xi^m)^{2(1-m)/m}]\} \\
& + [Z_3+Z_4](\xi,0)\frac{(1-m)}{m^2(2m-1)^3}\xi^{4(1-m)}\sum_{j=0}^{\infty}(-1)^j a_j\left(\frac{1-m}{2m-1}\right)^j\frac{1}{(j+1)}\times\left\{\left(\frac{\eta+\xi^m}{\xi^m}\right)^{j+1}F(2,\mu_j,\mu_j+1;-m^{-2}[\eta+\xi^m]^{2(1-m)/m})\right. \\
& \left. - F(2,\mu_j,\mu_j+1;-m^{-2}\xi^{2(1-m)})\right\} + \frac{2(1-m)}{m(2m-1)^2}\left[1-\frac{m^2}{\xi^{2(1-m)}}\right]Z_3(\xi,0)\xi^{2(1-m)}\sum_{j=0}^{\infty}(-1)^j\left(\frac{1-m}{2m-1}\right)^j\left\{\left[1-\frac{m^2}{2\xi^{2(1-m)}}\right]\right. \\
& \times\left[\left(\frac{\eta+\xi^m}{\xi^m}\right)^{j+1}F(1,\mu_j,\mu_j+1;-m^{-2}[\eta+\xi^m]^{2(1-m)/m}) - F(1,\mu_j,\mu_j+1;-m^{-2}\xi^{2(1-m)})\right] \\
& \left. -\frac{1}{2}\left[\left(\frac{\eta+\xi^m}{\xi^m}\right)^{j+1}F(2,\mu_j,\mu_j+1;-m^{-2}[\eta+\xi^m]^{2(1-m)/m}) - F(2,\mu_j,\mu_j+1;-m^{-2}\xi^{2(1-m)})\right]\right\} \\
& - \frac{2k^2(\xi-h)}{m^2(2m-1)^2}\left[1-\frac{m^4}{\xi^{4(1-m)}}\right]^{-1}\xi^{4-3m}\sum_{j=0}^{\infty}(-1)^j\left(\frac{1-m}{2m-1}\right)^j\left\{\left(\frac{\eta+\xi^m}{\xi^m}\right)^{j+1}F(1,\mu_j,\mu_j+1;-[1+2m^{-2}\xi^{2(1-m)}]\right. \\
& \left.\times[(\eta+\xi^m)/\xi^m]^{2(1-m)/m}) - F(1,\mu_j,\mu_j+1;-[1+2m^{-2}\xi^{2(1-m)}])\right\} + \frac{2}{m^2(2m-1)^2}\left[ik+\frac{m^2(1-m)}{\xi^{3-2m}}\right]\left[1-\frac{m^2}{\xi^{2(1-m)}}\right]\xi^{4-3m} \\
& \times\sum_{j=0}^{\infty}(-1)^j\left(\frac{1-m}{2m-1}\right)^j\times\left\{\left(\frac{\eta+\xi^m}{\xi^m}\right)^{j+1}F(1,\mu_j,\mu_j+1;-m^{-2}[\eta+\xi^m]^{2(1-m)/m}) - F(1,\mu_j,\mu_j+1;-m^{-2}\xi^{2(1-m)})\right\}
\end{aligned} \tag{21}$$

where the coefficients a_j and μ_j ($j \geq 0$) are defined by:

$$a_j = 1 + (3/2)j + (1/2)j^2$$

and

$$\mu_j = [m/2(1-m)](j+1)$$

Eq. (21) involves the unknown function f_{sn} through the quantities $Z_\nu(\xi,0)$. For the power-law shocks under consideration, these are given by:

$$\begin{aligned}
Z_1(\xi,0) = {} & -\left\{1-\frac{m^2}{\xi^{2(1-m)}}+0\left(\frac{1}{\xi^{4(1-m)}}\right)\right\}f'_{sn}(\xi) - \left\{ik-\frac{3(1-m)}{\xi}+\frac{4m^2(1-m)}{\xi^{3-2m}}+0\left(\frac{1}{\xi^{5-3m}}\right)\right\}f_{sn}(\xi) \\
& + ik\xi + ikm^2\xi^{2m-1} + (1-ikh) + 0\left(\frac{1}{\xi^2(1-m)}\right)
\end{aligned} \tag{22a}$$

‡In Ref. 8, Eq. (19d) should have a square bracket at the end of the third last line and the square bracket in the fourth last line should be].

$$Z_2(\xi,0)=\left\{\frac{1-m}{m\xi^m}+0\left(\frac{1}{\xi^{4(1-m)}}\right)\right\}f_{sn}(\xi)-\frac{2}{m}\left\{ik\xi^{2-m}+(1-ikh)\xi^{1-m}-ikm^4\xi^{3m-1}+0\left(\frac{1}{\xi^{1-m}}\right)\right\} \tag{22b}$$

$$Z_3(\xi,0)=\left\{\frac{m(1-m)}{\xi^{2-m}}+0\left(\frac{1}{\xi^{4-3m}}\right)\right\}f_{sn}(\xi)+ikm\xi^m+0\left(\frac{1}{\xi^{1-m}}\right) \tag{22c}$$

$$Z_3(\xi,0)+Z_4(\xi,0)=\left\{\frac{2(1-m)}{m\xi^m}+0\left(\frac{1}{\xi^{2-m}}\right)\right\}f_{sn}(\xi)+\frac{2ik}{m}\xi^{2-m}+\frac{2}{m}(1-ikh)\xi^{1-m}+0\left(\frac{1}{\xi^{1-m}}\right) \tag{22d}$$

As indicated earlier, the function f_{sn} can be determined by using Eqs. (21) and (22) in Eq. (20). This yields the ordinary differential equation[5]

$$\left\{b_1+\frac{b_2}{\xi^{2(1-m)}}+\ldots\right\}f'_{sn}+\left\{c_1+\frac{c_2}{\xi^{2(1-m)}}+\ldots\right\}f_{sn}=d_1\xi^2+d_2\xi^{2m}+\ldots \tag{23a}$$

where

$$b_1=\ln 2-\left(\frac{3m-1}{m}\right)\ln\left(\frac{m}{2m-1}\right)-\frac{(7m^4-3m^3-10m^2+9m-2)}{m^2(2m-1)^2}+\frac{2m(1-m)^2}{(2m-1)^3}$$
$$\times\sum_{j=0}^{\infty}(-1)^j a_j\left(\frac{1-m}{2m-1}\right)^j\frac{1}{(j+1)[j+2(3m-2)/m]}$$

$$c_1=-ik\left\{\frac{3-2m}{2-m}-\ln 2+\ln\frac{m}{2m-1}\right\}$$

$$d_1=-\frac{k^2(5m^2-8m+8)}{m(2m-1)(2-m)}+\frac{3k^2}{(2m-1)^2}\sum_{j=0}^{\infty}(-1)^j\left(\frac{1-m}{2m-1}\right)^j\frac{1}{[j+2(2m-1)/m]} \tag{23b}$$

Equations (23) are valid for large ξ (away from the nose) and hence the shock attachment condition, Eq. (9), cannot be applied. However, as determined in Ref. 8, in the asymptotic theory for the steady flow, the body and shock coincide up to a value of $x=x_0$, (see Eq. (18) of Ref. 8). One would therefore expect such attachment for the oscillatory flow as well. Hence, we replace Eq. (9) by

$$f_{sn}(x_0)=0 \tag{24}$$

The solution of Eq. (23a) subject to Eq. (24) is:

$$f_{sn}(\xi)=\frac{d_1}{c_1}\xi^2+\ldots \tag{25}$$

The pressure on the airfoil surface is obtained by using Eqs. (25) and (22) in Eqs. (21) together with $\eta=-\xi^m$ and expansions for the hypergeometric functions for large argument.[9] Hence,

$$Z_2(\xi,-\xi^m)=Z_2^{(1)}(\xi,-\xi^m)=C_1\xi^{3-m}+C_2\xi^{1+m}$$
$$+C_3\xi^{2-m}\ln\xi+0(\xi^{2-m})$$

where

$$C_1=\frac{-k^2}{m(2m-1)}$$

$$C_2=\frac{2m^2k^2}{(2m-1)^2}\sum_{j=0}^{\infty}(-1)^j\left(\frac{1-m}{2m-1}\right)^j\left(\frac{3-2\mu_j}{2-\mu_j}\right)$$

$$C_3=\frac{2(1-m)}{m}\left\{(1-m)e_0-2k\right\}i$$

with $ie_0=d_1/c_1$. Transforming via Eq. (13), the surface pressure may be written as

$$P|_{\text{body}}=P(x,\psi_{0n})|_{\psi_{0n}=0}=ReP(x,0)+iImP(x,0) \tag{26}$$

where

$$ReP(x,0)=-k^2m(2m-1)x^{3m-1}+0(x^{5m-3})$$

$$ImP(x,0)=2m(1-m)(2m-1)^2$$
$$\times\{(1-m)e_0-2k\}x^{3m-2}\ln x+0(x^{3m-2}) \tag{27}$$

B. Lift and Moments

The unsteady coefficient of lift is:

$$C_L=\theta(\bar t)L_1+\frac{\bar L}{k\bar U_\infty}\dot\theta(\bar t)L_2$$

where

$$\theta(\bar t)\equiv\alpha e^{ik\bar t},\quad \dot\theta(\bar t)\equiv\frac{d\theta}{d\bar t},\text{ and}$$

$$L_1=-4\int_0^1 ReP(x,0)\,dx=\frac{4}{3}k^2(2m-1)$$

and

$$L_2=-4\int_0^1 ImP(x,0)\,dx$$
$$=\frac{8m(1-m)(2m-1)^2}{(3m-1)^2}k\{(1-m)e_1-2\}$$

with $e_1=e_0/k$, so that e_1 is independent of k. The quantities L_1 and L_2 represent the in-phase and out-of-phase components of the lift coefficient.

The pitching moment coefficient has the form

$$C_m==\theta(\bar t)\{hL_1-M_1\}+\frac{\bar L}{k\bar U_\infty}\dot\theta(t)\{hL_2-M_2\}$$

Table 1 Damping- and restoring-moment coefficient for power-law bodies $y_{b0} \sim x^m$

h	m / k	0.79, 0.1	0.79, 0.5	0.75, 0.1	0.75, 0.5	0.70, 0.1	0.70, 0.5	0.67, 0.1	0.67, 0.5
0.0	m_θ	−.0027	−.0680	−.0023	−.0577	−.0018	−.0452	−.0015	−.0378
	$m_{\dot\theta}$	.1179		.1064		.0720		.0391	
0.1	m_θ	−.0023	−.0583	−.0020	−.0494	−.0015	−.0385	−.0013	−.0322
	$m_{\dot\theta}$	.0826		.0719		.0457		.0236	
0.2	m_θ	−.0019	−.0486	−.0016	−.0410	−.0013	−.0318	−.0011	−.0265
	$m_{\dot\theta}$	.0473		.0375		.0195		.0081	
0.3	m_θ	−.0016	−.0390	−.0013	−.0327	−.0010	−.0252	−.0008	−.0208
	$m_{\dot\theta}$	.0120		.0030		−.0067		−.0074	
0.4	m_θ	−.0012	−.0293	−.0010	−.0244	−.0007	−.0185	−.0006	−.0152
	$m_{\dot\theta}$	−.0232		−.0315		−.0330		−.0228	
0.5	m_θ	−.0008	−.0196	−.0006	−.0160	−.0005	−.0118	−.0004	−.0095
	$m_{\dot\theta}$	−.0585		−.0660		−.0592		−.0383	
0.6	m_θ	−.0004	−.0100	−.0003	−.0077	−.0002	−.0052	−.0002	−.0038
	$m_{\dot\theta}$	−.0938		−.1004		−.0854		−.0538	
0.7	m_θ	−.0000	−.0003	.0000	.0006	.0001	.0015	.0001	.0018
	$m_{\dot\theta}$	−.1290		−.1350		−.1117		−.0693	
0.8	m_θ	.0004	.0094	.0004	.0090	.0003	.0082	.0003	.0075
	$m_{\dot\theta}$	−.1642		−.1694		−.1379		−.0848	
0.9	m_θ	.0008	.0190	.0007	.0173	.0006	.0148	.0005	.0132
	$m_{\dot\theta}$	−.1996		−.2039		−.1641		−.1002	
1.0	m_θ	.0011	.0287	.0010	.0256	.0009	.0215	.0008	.0188
	$m_{\dot\theta}$	−.2348		−.2384		−.1904		−.1157	

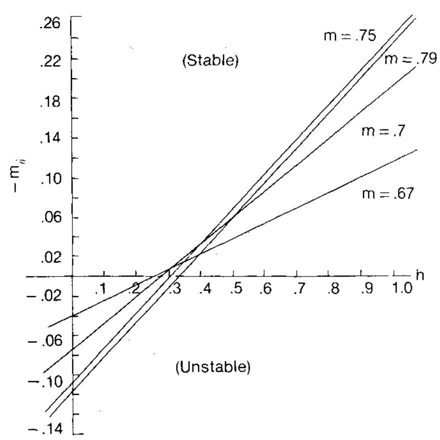

Fig. 1 Damping-moment coefficient, $-m_{\dot\theta}$, vs pivot position, h, for various m and arbitrary k (see Eq. (28)).

where

$$M_1 = -4\int_0^1 x\,Re P(x,0)\,dx = \frac{4m(2m-1)}{3m+1}k^2$$

and

$$M_2 = -4\int_0^1 x\,Im P(x,0)\,dx$$

$$= \frac{8(1-m)(2m-1)^2}{9m}\{(1-m)e_1 - 2\}k$$

Here, M_1 and M_2 are referred to as the in-phase and out-of-phase components of the pitching moment.

Defining the restoring- and damping-moment coefficients m_θ and $m_{\dot\theta}$ as in Ref. 10, we find

$$-m_\theta = -\frac{2(2m-1)}{3}k^2 h + \frac{2m(2m-1)}{3m+1}k^2 \tag{28a}$$

and

$$-m_{\dot\theta} = 4(1-m)(2m-1)^2 \times \left\{\frac{m}{(3m-1)^2}h - \frac{1}{9m}\right\}\{2-(1-m)e_1\} \tag{28b}$$

C. Numerical Results and Discussion

Table 1 constitutes numerical values for m_θ and $m_{\dot\theta}$ against h for $k=0.1$ and 0.5 and for several values of the power m of the body shape. Equations (28) show that both m_θ and $m_{\dot\theta}$ are linear functions of h while m_θ is quadratic in k and $m_{\dot\theta}$ is independent of k. The damping-moment coefficient becomes zero at $h=(3m-1)^2/(9m^2)$. Figure 1 shows that stability is enhanced by placing the pivot position behind this value of h and is decreased for pivot positions ahead of it.

References

[1] McIntosh, Jr., S. C., "Studies in Unsteady Hypersonic Flow Theory," Ph.D. Dissertation, Stanford University, Aug. 1965.

[2] Guiraud, J. P., Vallée, D., and Zolver, R., "Bluntness Effects in Hypersonic Small-Disturbance Theory," *Basic Developments in Fluid Dynamics*, Vol. 1, Academic Press, N.Y., London, 1965.

[3] Barron, R. M., "Oscillating Airfoils: I. Wedges of Arbitrary Thickness in Supersonic and Hypersonic Flow," *AIAA Journal*, to be published.

[4]Cole, J. D., "Newtonian Flow Theory for Slender Bodies," *Journal of Aerospace Sciences*, Vol. 24, June 1957, pp. 448-455.

[5]Barron, R. M., "Unsteady Newtonian Flow Past Oscillating Bodies with Application to Airfoils Supporting Power-Law Shocks," Ph.D. Dissertation, Carleton University, June 1974.

[6]Noble, B., *Applied Linear Algebra*, Prentice-Hall, N.J., 1969, pp. 361-364.

[7]Garabedian, P. R., *Partial Differential Equations*, Wiley and Sons Inc., New York, London, Sydney, 1964.

[8]Barron, R. M., "On Newtonian Flow Past Power-Law Bodies," *AIAA Journal*, Vol. 15, Jan. 1977, pp. 117-119.

[9]Whittaker, E. T. and Watson, G. N., *Modern Analysis*, Cambridge University Press, New York, 1935.

[10]Van Dyke, M. D., "Supersonic Flow Past Oscillating Airfoils Including Nonlinear Thickness Effects," NACA Rept. No. 1183, 1954.